On the Anelastic Approximation for Deep Convection

FRANK B. LIPPS

Geophysical Fluid Dynamics Laboratory/NOAA, Princeton University, Princeton, New Jersey

(Manuscript received 3 October 1989, in final form 28 February 1990)

ABSTRACT

A brief review of the scale analysis of Lipps and Hemler is given without any reference to the parameters G and B. The resulting anelastic equations conserve energy, in contrast to the modified anelastic set of equations analyzed by Durran. In addition, the present equations give an accurate solution for the frequency of gravity waves in an isothermal atmosphere. The present anelastic equations have these characteristics in common with the pseudo-incompressible equations introduced by Durran.

The equations obtained from the scale analysis are appropriate for numerical integration of deep convection. The associated Poisson equation can be solved using standard procedures. For the pseudo-incompressible set of equations, the Poisson equation is more difficult to solve.

1. Introduction

The recent paper by Durran (1989) has brought on a renewal of my interest in the scale analysis of Lipps and Hemler (1982, 1985) for deep moist convection in the troposphere. One drawback of those studies was the undue emphasis placed on assigning values for the constants G and B when the scale analysis can only infer that these parameters are the order of one. For this reason, a brief review of the scale analysis for the deep anelastic equations is given without any reference to G and B. The results of this scale analysis are then discussed in the context of the conclusions of Durran (1989).

A major motivation for this study is to show that the Lipps and Hemler (1982) set of anelastic equations are superior to the modified anelastic equations analyzed by Durran (1989). The latter equations were originally formulated by Wilhelmson and Ogura (1972) and have been used by many numerical modelers since then. As shown by Durran, the modified anelastic equations do not conserve energy and give a poor representation of the frequency of gravity waves in an isothermal atmosphere. In contrast, the present set of equations does conserve energy for adiabatic frictionless flow. The analysis presented below indicates that these equations also give an accurate solution for the frequency of gravity waves in an isothermal atmosphere. In these respects the present set of equations has much in common with the pseudo-incompressible equations introduced by Durran (1989).

Corresponding author address: Dr. Frank B. Lipps, Geophysical Dynamics Laboratory, Princeton University, P.O. Box 308, Princeton, New Jersey 08542.

2. Review of the scale analysis

a. The scale analysis assumptions

A detailed discussion of the scale analysis assumptions is given in Lipps and Hemler (1982, hereafter referred to as LH82). It is not intended to redo the full discussion in that study but rather to emphasize the main points. It is hoped that this approach will clarify the essential elements of the scale analysis. In this discussion all variables are dimensional.

The first assumption is that all of the thermodynamic variables are separated into a base state part and a convective part which is the order of ϵ smaller. In particular, for the potential temperature θ:

$$\theta = \theta_0(z) + \theta_1(x, y, z, t). \tag{1}$$

Defining θ_{00} as $\theta_0(0)$ and $\Delta\theta_1$ as the maximum value of θ_1, then ϵ is given by

$$\frac{\Delta\theta_1}{\theta_{00}} = \epsilon \ll 1. \tag{2}$$

The second assumption is applied to the first law of thermodynamics, which is written as

$$\underset{\text{(a)}}{\frac{d\theta_1}{dt}} + \underset{\text{(b)}}{w\frac{d\theta_0}{dz}} = \underset{\text{(c)}}{\frac{H}{c_p\rho_0\pi_0}} \tag{3}$$

where H is the net heating introduced by Durran (1989). For clarity in comparison with his study, the base state density ρ_0, Exner pressure function π_0 and potential temperature θ_0 are represented by $\bar{\rho}$, $\bar{\pi}$ and $\bar{\theta}$ in his analysis. The three terms in the above equation are labeled as (a), (b) and (c). In LH82 the assumption

(a) ~ (b) is made. For deep moist convection this assumption implies:

$$(a) \sim (b) \sim (c). \tag{4}$$

Using the characteristic length scale l and the velocity scale W, (a) has the magnitude

$$\frac{d\theta_1}{dt} \sim \theta_{00} \frac{W}{l} \epsilon \tag{5}$$

where we have implied that the time scale τ is equal to l/W. Now setting (b) ~ (a) we obtain

$$\frac{1}{\theta_0} \frac{d\theta_0}{dz} \sim \frac{\epsilon}{l}. \tag{6}$$

The equation (6) represents a key conclusion of the scale analysis in LH82. From a physical point of view, this relation indicates that the vertical gradient of the base state θ_0 is the same order of magnitude as the vertical (or horizontal) gradient of the disturbance θ_1. Since the length scale l is considered finite, the base state potential temperature θ_0 is a slowly varying function of z as stated in LH82.

The assumption $\epsilon \ll 1$ is not sufficient to remove acoustic waves from the equations of motion. As discussed by Ogura and Phillips (1962), for the anelastic equations it is necessary to assume in addition that the time scale τ is set by the inverse of the Brunt–Väisälä frequency N. Thus, as given in LH82, this assumption is represented by

$$\tau \sim N^{-1}, \quad N^2 = \frac{g}{\theta_{00}} \frac{\Delta\theta_T}{d} \tag{7}$$

where $\Delta\theta_T$ is the total change in the base state θ_0 through the depth d of the troposphere.

Finally it is shown that (6) and (7) together imply nonhydrostatic convection. First, we note the relations

$$\frac{1}{\theta_0} \frac{d\theta_0}{dz} \approx N^2/g, \quad \tau = l/W \tag{8}$$

where N^2 is defined by (7). Now using (6), (8) and $\tau \sim N^{-1}$ we find

$$g\epsilon l/W^2 \sim 1. \tag{9}$$

Thus (9) is equivalent to the second part of (16) in LH82. Since the time derivative of vertical velocity is scaled by W^2/l and the buoyancy by $g\epsilon$, it is seen that these two terms in the w-momentum equation are the same order of magnitude, implying nonhydrostatic convection. Thus moist convection on the scales of mesoα and mesoβ (Fujita 1963; Orlanski 1975) is formally excluded from the present scale analysis.

In retrospect, the exclusion of larger-scale hydrostatic, convectively-driven phenomena from this scale analysis appears unduely restrictive. A further discussion of this topic will be given in the final section of this paper.

b. The continuity equation

In this section the continuity equation in LH82 is obtained by starting from the more general equation given by Durran (1989).

The pseudo-incompressible continuity equation is given by Eq. (7) in his study:

$$\frac{w}{\rho_0\theta_0} \frac{d\rho_0\theta_0}{dz} + \nabla \cdot \mathbf{V} = \frac{H}{c_p\rho_0\theta_0\pi_0}. \tag{10}$$

This equation can be written as

$$\nabla \cdot \mathbf{V} + w\left(\frac{1}{\rho_0} \frac{d\rho_0}{dz} + \frac{1}{\theta_0} \frac{d\theta_0}{dz}\right) = \frac{H}{c_p\rho_0\theta_0\pi_0}. \tag{10a}$$

But from Eq. (3) in the present study we see that this equation can be simplified to

$$\nabla \cdot \mathbf{V} + w \frac{1}{\rho_0} \frac{d\rho_0}{dz} - \frac{1}{\theta_0} \frac{d\theta_1}{dt} = 0. \tag{10b}$$

In order to evaluate the relative magnitude of the last term in (10b), it is compared with the $\partial w/\partial z$ component of the total divergence $\nabla \cdot \mathbf{V}$. Using (5) to evaluate $\theta_0{}^{-1} d\theta_1/dt$ we find

$$\frac{\partial w}{\partial z} \sim W/l, \quad \frac{1}{\theta_0} \frac{d\theta_1}{dt} \sim (W/l)\epsilon. \tag{11}$$

Thus, the last term in (10b) is small so that to leading order:

$$\nabla \cdot \mathbf{V} + w \frac{1}{\rho_0} \frac{d\rho_0}{dz} = 0 \tag{12}$$

which is the continuity equation in LH82. Therefore, this equation can be obtained by starting from (7) in Durran and applying consistently the LH82 scale analysis.

To complete the discussion of the continuity equation, it is relevant to note the magnitude of the $wd(\ln\rho_0)/dz$ term. Using the solution in an isentropic atmosphere as a guide

$$\frac{1}{\rho_0} \frac{d\rho_0}{dz} \sim \frac{c_v}{R} \frac{1}{H} \tag{13}$$

where $H = c_p\theta_{00}/g \approx 30$ km and $c_v/R = 2.5$. Thus, calculating the ratio

$$R_1 = \left| \frac{wd(\ln\rho_0)/dz}{\partial w/\partial z} \right| \sim 2.5 \frac{l}{H}. \tag{14}$$

Since the length scale l is associated with derivatives, it is appropriate to think of it in terms of an inverse wavenumber. If we consider a 10 km deep convective cell, then $l = 10$ km/(3.14) is an appropriate length scale. Since $H \approx 30$ km, (14) becomes

$$R_1 \sim \frac{2.5}{3.14} \frac{1}{3} = 0.265. \tag{14a}$$

Thus, the density gradient term is small but not negligible compared with $\partial w/\partial z$.

c. The pressure gradient term

A key result of the LH82 scale analysis is that the vertically varying base state potential temperature θ_0 can be pulled inside the pressure gradient term. Using tensor notation

$$-c_p\theta_0\frac{\partial\pi_1}{\partial x_i} = -\frac{\partial}{\partial x_i}(c_p\theta_0\pi_1) + c_p\pi_1\frac{d\theta_0}{dz}\delta_{i3}. \quad (15)$$

For the vertical momentum equation we find that ratio

$$R_2 = \left|\frac{c_p\pi_1\dfrac{d\theta_0}{dz}}{\partial(c_p\theta_0\pi_1)/\partial z}\right| \sim \frac{l}{\theta_0}\frac{d\theta_0}{dz} \sim \epsilon \quad (16)$$

where the relation (6) has been used to obtain $R_2 \sim \epsilon$. Thus, the term involving $d\theta_0/dz$ in (15) can be neglected to leading order. For this reason it is appropriate to define the pressure variable:

$$\phi \equiv c_p\theta_0\pi_1. \quad (17)$$

Hence gradients of ϕ correspond to the pressure gradient terms in LH82.

3. Energetics

A major consequence of not having $\theta_0(z)$ as a multiple of the pressure gradient terms is that energy conservation exists for the LH82 equations. Following Durran (1989) we first consider the two-dimensional linear perturbation equations for frictionless adiabatic flow.

$$\frac{\partial u'}{\partial t} + \frac{\partial \phi'}{\partial x} = 0 \quad (18)$$

$$\frac{\partial w'}{\partial t} + \frac{\partial \phi'}{\partial z} = g\frac{\theta'}{\theta_0} \quad (19)$$

$$\frac{\partial \theta'}{\partial t} + \frac{\theta_0}{g}N_0^2 w' = 0 \quad (20)$$

$$\frac{\partial \rho_0 u'}{\partial x} + \frac{\partial \rho_0 w'}{\partial z} = 0 \quad (21)$$

where (18) and (19) are the momentum equations, (20) is the thermodynamic equation and (21) is the continuity equation. Here we have defined

$$N_0^2 = \frac{g}{\theta_0}\frac{d\theta_0}{dz}. \quad (22)$$

Thus, in general, N_0^2 is a function of z whereas N^2 as defined by (7) for the scale analysis is a constant.

The equation for the total perturbation energy can be obtained in a straightforward manner from (18)–(21):

$$\frac{\partial E'}{\partial t} + \frac{\partial p'u'}{\partial x} + \frac{\partial p'w'}{\partial z} = 0 \quad (23)$$

where E' is the total perturbation energy

$$E' = \frac{\rho_0}{2}\left(u'^2 + w'^2 + \frac{g^2}{N_0^2}\frac{\theta'^2}{\theta_0^2}\right) \quad (24)$$

and $p' = \rho_0\phi'$. It is seen that these equations correspond to (49) and (51) of Durran (1989). If θ_0 had been a multiple of the pressure gradient terms in (18) and (19), then Durran's Eq. (52) would have been obtained, which is similar to (23) but has the term $p'w'd\ln\partial_0/dz$ on the right. This nonconservative term is not present for the LH82 equations or the pseudo-incompressible equations. It is, however, present for the modified anelastic equations as discussed by Durran. Thus, evidently both Durran's pseudo-incompressible equations and the LH82 equations eliminate the energetic inconsistency noted by Wilhelmson and Ogura (1972).

For three-dimensional finite amplitude flow we find a form of energy conservation similar to that of Durran. If the total pressure is defined as $p^* \equiv \rho_0 R\pi_0\theta_0 + \rho_0\phi$, the total energy equation may be written in the conservative form:

$$\frac{\partial E_{\rm LH}}{\partial t} + \nabla\cdot[(E_{\rm LH} + p^*)\mathbf{V}] = 0 \quad (25)$$

where

$$E_{\rm LH} = \rho_0\left(\frac{u^2+v^2+w^2}{2} + c_p\pi_0\theta_1 + gz\right) + c_v\rho_0T_0. \quad (26)$$

Since $c_p\pi_0\theta_1 = c_pT_1$ for the LH82 scale analysis, the presence of θ_1 comes in through the first-order sensible heat. In Durran (1989), evidently the effect of θ_1 comes in through the $\rho^* gz$ term in his Eq. (60).

4. Gravity waves in an isothermal atmosphere

In this section the frequency of gravity waves in an isothermal atmosphere is calculated for the LH82 perturbation equations (18)–(21).

Following Durran, we remove the effect of the decrease in density with height by defining the new variables

$$\tilde u = \left(\frac{\rho_0}{\rho_{00}}\right)^{1/2}u', \quad \tilde w = \left(\frac{\rho_0}{\rho_{00}}\right)^{1/2}w',$$

$$\tilde\phi = \left(\frac{\rho_0}{\rho_{00}}\right)^{1/2}\phi'; \quad \tilde\theta = \left(\frac{\rho_0}{\rho_{00}}\right)^{1/2}\frac{g}{\theta_0}\theta' \quad (27)$$

where ρ_{00} is a constant reference density. In terms of these new variables, Eqs. (18)–(21) can be written as

$$\frac{\partial \tilde u}{\partial t} + \frac{\partial \tilde\phi}{\partial x} = 0 \quad (28)$$

$$\frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{\phi}}{\partial z} + \beta_0 \tilde{\phi} = \tilde{\theta} \quad (29)$$

$$\frac{\partial \tilde{\theta}}{\partial t} + N_0^2 \tilde{w} = 0 \quad (30)$$

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{w}}{\partial z} - \beta_0 \tilde{w} = 0 \quad (31)$$

where

$$\beta_0 = -\frac{1}{2} \frac{1}{\rho_0} \frac{d\rho_0}{dz}. \quad (32)$$

Comparing with Durran, it is seen that (28)–(31) are isomorphic with his (33)–(36) when $\delta_1 = \delta_2 = 0$ and his parameter Γ is replaced by β_0. Thus, it is evident that gravity waves described by the above equations will have many properties in common with gravity waves obtained from the pseudo-incompressible equations (which correspond to $\delta_1 = \delta_2 = 0$ in Durran's analysis).

In the case of an isothermal atmosphere, N_0^2 and β_0 are constant and solutions may be obtained of the form

$$(\tilde{u}, \tilde{w}, \tilde{\theta}, \tilde{\phi}) = (\hat{u}, \hat{w}, \hat{\theta}, \hat{\phi}) \exp[i(kx + mz - \omega t)]. \quad (33)$$

Since (28)–(31) are isomorphic with Durran's equations for the pseudo-incompressible case, it follows that the frequency is given by Eq. (43) in Durran with Γ replaced by β_0. Thus, for the LH82 scale analysis

$$\omega_{\mathrm{LH}}^2 = \frac{N_0^2 k^2}{k^2 + m^2 + \beta_0^2}. \quad (34)$$

For the modified anelastic equations Durran found that ω^2 was complex. In the present case, since energy conservation applies to the LH82 equations, ω_{LH}^2 is real as expected.

At this point it is necessary to relate β_0 with the parameters used by Durran for an isothermal atmosphere. Equation (37) in Durran can be written as

$$\Gamma = -\beta_0 + \frac{g}{c_s^2} = \beta_0 - \frac{N_0^2}{g} \quad (35)$$

where c_s is the speed of sound. Using the second equality in (35) gives

$$\beta_0 = \Gamma + \frac{N_0^2}{g}. \quad (36)$$

Now using both equalities in (35), after some algebra, it can be shown

$$\beta_0^2 = \Gamma^2 + N_0^2/c_s. \quad (37)$$

Thus, the equation for ω_{LH}^2 becomes

$$\omega_{\mathrm{LH}}^2 = \frac{N_0^2 k^2}{k^2 + m^2 + \Gamma^2 + N^2/c_s^2}. \quad (38)$$

Hence the solution for ω_{LH}^2 is equivalent to the more accurate solution in Eq. (42) of Durran, which was obtained with the effect of sound waves included ($\delta_1 = 1$). The Eq. (43) in Durran does not include the N_0^2/c_s^2 term in the denominator. This degree of accuracy of (38) is fortuitous[1] since the scale analysis of LH82 has its primary validity for the troposphere.

It should also be noted that (38) is only slightly more accurate than Eq. (43) in Durran, which was obtained from the pseudo-incompressible equations. The maximum increase in accuracy in using (38) is for long waves ($k^2 \to 0$) and for relatively deep disturbances in the vertical. Thus, using $k^2 \to 0$ and a vertical wavelength of 10 km, ω_{LH}^2 as given by (38) is one percent smaller than the corresponding value of ω^2 calculated from Eq. (43) in Durran. This conclusion is consistent with the discussion given by Durran in comparing the solutions from his Eq. (42) and (43).

5. Summary and conclusions

A review of the scale analysis of LH82 for deep moist convection in the troposphere has been presented. In this scale analysis all thermodynamic variables are separated into a base state part and a convective part which is the order of ϵ smaller. It is assumed that $\epsilon \ll 1$. The second assumption, which appears to be the unique aspect of LH82, is that the first two terms in the thermodynamic equation (3) are the same order of magnitude. It is this assumption which leads to the conclusion that the base state potential temperature θ_0 is a slowly varying function of z.

The continuity equation (12) is obtained from Durran's more general form of continuity by a consistent use of the LH82 scale analysis. In addition, by applying the second assumption, it is shown that the height-dependent θ_0 can be put inside the pressure gradient terms. When written in this form, the LH82 set of equations conserve energy, thus eliminating the energetic inconsistency noted by Wilhelmson and Ogura (1972). The energy conservation properties of the LH82 equations and the pseudo-incompressible equations of Durran (1989) appear to be very similar.

Since the present scale analysis is based on the assumption that the horizontal and vertical length scales are the same, it was found that the convection is nonhydrostatic. In retrospect this conclusion appears overly restrictive. When the flow is hydrostatic, the horizontal length scale l_x is much greater than the vertical length scale l_z, which is roughly equivalent in magnitude to the present length scale l. Thus, making the second assumption, we can again derive (6) but with l replaced by l_z. The argument for putting the height dependent θ_0 inside the pressure gradient terms follows as before with the resultant conservation of energy. For hydro-

[1] If sound waves are included in (28)–(31), then (38) contains an *extra* N_0^2/c_s^2 term in the denominator.

static flow with $l_x \gg l_z$, the time scale τ for internal gravity waves is obtained from (38) with $k^2 \ll m^2$. Since the term $\Gamma^2 + N^2/c_s^2$ in the denominator of (38) is very small, we find $\tau \sim (Nk)^{-1}m$ or $\tau \sim Nl_x l_z^{-1}$ as the required condition that acoustic waves be excluded. The resultant large time scale (low frequency) is consistent with the flow being hydrostatic. Thus it appears that, with a slight generalization of the discussion, the present scale analysis equations are valid for hydrostatic dynamics as well.

An important conclusion of this study is that the LH82 anelastic equations have characteristics superior to the modified anelastic equations analyzed by Durran (1989). The modified anelastic equations give complex frequencies for gravity waves in an isothermal atmosphere. This result is physically unrealistic and is associated with the lack of energy conservation for these equations. In contrast, the LH82 equations conserve energy and give accurate real frequencies for such internal gravity waves.

The above results indicate that the LH82 anelastic equations have much in common with the pseudo-incompressible equations introduced by Durran (1989). An advantage of the latter equations is that the pseudo-incompressible equations are valid over the total atmosphere whereas the LH82 equations are rigorously valid only in the troposphere. The assumption, given by Eq. (6), that θ_0 is a slowly varying function of z is highly suspect for the stratosphere. In spite of this shortcoming, the present analysis suggests that gravity waves in the stratosphere are adequately represented by the LH82 equations.

Thus the LH82 anelastic equations are appropriate for numerical integration of deep convection. The associated Poisson equation can be solved using standard procedures. In contrast, there are difficulties involved in solving the Poisson equation for the pseudo-incompressible system (Durran 1989). Thus the latter equations have the advantage of a rigorous formulation for the stratosphere but present more difficulty in numerical integration.

Acknowledgments. My special thanks go to Richard Hemler and Dale Durran who made several helpful comments in regards to this study. The manuscript was typed by Wendy Marshall.

REFERENCES

Durran, D. R., 1989: Improving the anelastic approximation. *J. Atmos. Sci.*, **46**, 1453–1461.

Fujita, T., 1963: Analytical mesometeorology: A review. *Severe Local Storms, Meteor. Monogr.*, **5**, No. 26, Amer. Meteor. Soc., 77–125.

Lipps, F., and R. Hemler, 1982: A scale analysis of deep moist convection and some related numerical calculations. *J. Atmos. Sci.*, **39**, 2192–2210.

——, and ——, 1985: Another look at the scale analysis for deep moist convection. *J. Atmos. Sci.*, **42**, 1960–1964.

Orgura, Y., and N. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.*, **19**, 173–179.

Orlanski, I., 1975: A rational subdivision of scales for atmospheric processes. *Bull. Amer. Meteor. Soc.*, **56**, 527–530.

Wilhelmson, R., and Y. Ogura, 1972: The pressure perturbation and the numerical modeling of a cloud. *J. Atmos. Sci.*, **29**, 1295–1307.